Hamiltonian of a homogeneous two-component plasma

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The Hamiltonian of one- and two-component plasmas is calculated in the negligible radiation Darwin approximation. Since the Hamiltonian is the phase space energy of the system its form indicates, according to statistical mechanics, the nature of the thermal equilibrium that plasmas strive to attain. The main issue is the length scale of the magnetic interaction energy. In the past a screening length $\lambda = 1/\sqrt{r_e n}$, with n number density and r_e classical electron radius, has been derived. We address the question whether the corresponding longer screening range obtained from the classical proton radius is physically relevant and the answer is affirmative. Starting from the Darwin Lagrangian it is nontrivial to find the Darwin Hamiltonian of a macroscopic system. For a homogeneous system we resolve the difficulty by temporarily approximating the particle number density by a smooth constant density. This leads to Yukawa-type screened vector potential. The nontrivial problem of finding the corresponding, divergence free, Coulomb gauge version is solved.

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I. INTRODUCTION

Plasmas are rarely in thermal equilibrium since both natural and human made plasmas almost always have large temperature, density, or pressure gradients. Nevertheless, plasmas, just as other forms of macroscopic matter, have a strong tendency towards thermal equilibrium. It would therefore be of interest to know what this equilibrium state is like. According to statistical mechanics the phase space probability distribution is given by the canonical Maxwell-Boltzmann distribution

$$f(\boldsymbol{r},\boldsymbol{p}) = \frac{1}{Z}\exp\left(-\frac{H(\boldsymbol{r},\boldsymbol{p})}{k_B T}\right) \tag{1}$$

and thus the key to the equilibrium distribution is the Hamiltonian $H(\boldsymbol{r},\boldsymbol{p}) = E$, or phase space energy, of the system. A conserved energy for a system of charged particles exists only in the, so called, Darwin approximation. Since radiation is a higher order process it is probably not a bad approximation to consider an equilibrium plasma as an equilibrium between a Darwin charged particle gas and a Planck distributed, black body, photon gas.

Plasma physics has been approached from the Darwin approximation point of view by several authors. Simulations and numerical studies based on the Darwin Lagrangian have been quite successful [1–6]. In particular Mehra and De Luca [7] have shown using simulations based on the Darwin Lagrangian that long range magnetic order may arise. Attempts to use statistical mechanics and the Darwin Hamiltonian have been hampered by the fact that the Darwin Hamiltonian for macroscopic systems is not known explicitly, but there have been several noteworthy attempts [8–16].

An important step was taken by Jones and Pytte [17] who derived an approximate Hamiltonian for a one-component homogeneous plasma and found that the magnetic interaction energy is screened with the characteristic length scale $\lambda = 1/\sqrt{r_e n}$, where n is particle number density and r_e is classical electron radius. Jones and Pytte started from quantum mechanics and worked in $\boldsymbol{k}$ space (or momentum space, since $\boldsymbol{p} = \hbar\boldsymbol{k}$). Similar results were found by Essén [18,19] who showed that the classical Darwin Lagrangian for a homogeneous plasma should give rise to a Yukawa screened magnetic interaction energy in the Hamiltonian. The fact that the vector potential in the Darwin formalism must be divergence free, or transverse (Coulomb gauge) has, however, not been properly handled before, but here it will be.

II. OVERVIEW OF CONTENTS

We first review the fundamental status of the Darwin Lagrangian. This is needed because several important aspects of the Darwin approximation do not appear to be well known. The even more problematic and unknown status of the Darwin Hamiltonian is then discussed.

Since it is easy to find an energy, expressed in terms of the canonical momenta, but with the vector potential still given in terms of velocities, the problem of finding the Hamiltonian is reduced to expressing the vector potential in terms of the momenta. We therefore then discuss how we find the vector potential in terms of the canonical momenta and how to make it transverse. This is necessary since the Coulomb gauge is essential in the Darwin approximation.

The vector potential in terms of the canonical momenta is found using our main approximation: the real particle number density, a sum of δ functions, is replaced by a smooth constant density. We do this first for a one-component plasma (of either electrons or ions), then for a two-component plasma consisting of positive ions and negative electrons. Finally we consider the problem of how these are related. The outcome indicates that the length scale corresponding to the ion mass should be of relevance for plasma structures.

III. THE DARWIN LAGRANGIAN

The Darwin [20] negligible radiation approximation to electrodynamics is often presented as an expansion in v/c

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where terms up to $(v/c)^2$ are retained [21–26]. This gives correct results but is a bit puzzling since radiation is due to acceleration, not velocity itself. In fact, the exact retarded Lienard-Wiechert potentials show that the electromagnetic field from moving charges can be split into bound (or velocity) fields and radiation (or acceleration) fields [27,28]. The latter are normally much smaller than the velocity fields. A more fundamental way of viewing the approximation is to see it as neglect of the acceleration dependent part of the fields [19]. Once acceleration is neglected all retardations can be calculated and even the relativistic Lagrangian can be expressed in terms of particle positions and velocities at some given time.

The magnetic interaction energy found by Darwin was in fact derived earlier by Heaviside [29], for a historical discussion, see Jackson and Okun [30]. The same interaction was derived from the Dirac equation by Breit [31,32]. For other fundamental aspects of this approach see, for example, Sucher [33] or Moore *et al.* [34,35].

Magnetism is of importance in physics either because velocities are large compared to the speed of light or because a large, possibly macroscopic, number of particles compensate for the smallness of v/c (see Rindler [36], Sec. 7.7, for a discussion). Here we will be interested in the latter case. This means that the relativistic correction to the kinetic energy can be neglected. In small systems this term is inevitably of the same importance as magnetic effects but not in macroscopic ones (since it is a one-particle quantity it will not change the qualitative dynamics of the system). The use of the Darwin Lagrangian for macroscopic systems has been discussed, from a fundamental point of view, by Coleman and Van Vleck [37].

Consider a system of N charged particles with masses m_i, charges q_i, positions $\boldsymbol{r}_i$, and velocities $\boldsymbol{v}_i$. When we neglect the relativistic correction to the kinetic energy the Darwin Lagrangian of the system is

$$L(\boldsymbol{r},\boldsymbol{v})=\sum_{i=1}^{N}\left(\frac{1}{2}m_i\boldsymbol{v}_i^2+\frac{q_i}{2c}\boldsymbol{v}_i\cdot\boldsymbol{A}^c(\boldsymbol{r}_i)-\frac{q_i}{2}\phi(\boldsymbol{r}_i)\right), \quad (2)$$

where

$$\phi(\boldsymbol{r})=\sum_j^N\frac{q_j}{|\boldsymbol{r}-\boldsymbol{r}_j|}, \quad (3)$$

and, with $\boldsymbol{e}_j\equiv(\boldsymbol{r}-\boldsymbol{r}_j)/|\boldsymbol{r}-\boldsymbol{r}_j|$,

$$\boldsymbol{A}^c(\boldsymbol{r})=\sum_j^N\frac{\boldsymbol{v}_j+(\boldsymbol{v}_j\cdot\boldsymbol{e}_j)\boldsymbol{e}_j}{2c}\frac{q_j}{|\boldsymbol{r}-\boldsymbol{r}_j|}. \quad (4)$$

In $\phi(\boldsymbol{r}_i)$ and $\boldsymbol{A}^c(\boldsymbol{r}_i)$ the infinite terms arising from $i=j$ are, of course, excluded.

The generalized momentum $\boldsymbol{p}_i=\partial L/\partial\boldsymbol{v}_i$ is given by

$$\boldsymbol{p}_i=m_i\boldsymbol{v}_i+\frac{q_i}{c}\boldsymbol{A}^c(\boldsymbol{r}_i). \quad (5)$$

We note that this is the same expression as the one obtained in the familiar case of an external magnetic field. The factor 1/2 in front of the magnetic interaction term in Eq. (2) disappears in the differentiation since the magnetic interaction term is quadratic in the velocities. Note that the Coulomb gauge ($\boldsymbol{\nabla}\cdot\boldsymbol{A}^c=0$), which is used in the derivation of the Darwin Lagrangian, is essential for this result since it is only in this gauge that the Coulomb potential is velocity independent.

Using this one can find [19,38] the following expression for the energy:

$$E=\sum_{i=1}^{N}\left(\frac{\boldsymbol{p}_i^2}{2m_i}-\frac{q_i}{2m_ic}\boldsymbol{p}_i\cdot\boldsymbol{A}^c(\boldsymbol{r}_i)+\frac{q_i}{2}\phi(\boldsymbol{r}_i)\right). \quad (6)$$

Note that this is not yet the Hamiltonian since $\boldsymbol{A}^c(\boldsymbol{r}_i)$ is still expressed only in terms of velocities through Eq. (4). Also note the absence of an $\boldsymbol{A}^2$ term, which is due to the fact that we are not dealing with an external field but with the vector potential from the particles of the system itself.

It is possible to derive an exact expression for the energy from the Lagrangian (2) in terms of positions and velocities only. One finds [38,39,26,40] that the energy is given by

$$E(\boldsymbol{r},\boldsymbol{v})=\sum_{i=1}^{N}\frac{1}{2}m_i\boldsymbol{v}_i^2+\sum_{i<j}^{N}\frac{q_iq_j}{r_{ij}}+\sum_{i<j}^{N}\frac{q_iq_j}{2c^2}\frac{[\boldsymbol{v}_i\cdot\boldsymbol{v}_j+(\boldsymbol{v}_i\cdot\boldsymbol{e}_{ij})(\boldsymbol{v}_j\cdot\boldsymbol{e}_{ij})]}{r_{ij}} \quad (7)$$

(here $r_{ij}=|\boldsymbol{r}_i-\boldsymbol{r}_j|$). This expression, unfortunately, is misleading since it seems to predict that parallel currents raise the energy, in stark contradiction to the well known fact that parallel currents attract (see Schwinger *et al.* [26] for a discussion). The lesson to be learned from this is that the energy must be expressed in phase space (positions and canonical momenta) in order to give physically sensible results. We therefore now proceed to find the Hamiltonian corresponding to the Darwin Lagrangian of Eq. (2).

IV. THE DARWIN HAMILTONIAN

If we assume that there are N particles in the system and introduce generalized coordinates $Q^a, a=1,\ldots,3N$, according to $Q^1=x_1=\boldsymbol{r}_1\cdot\boldsymbol{e}_x,\ldots,Q^{3N}=z_N$, and corresponding generalized velocities, $\dot{Q}^a$, given by $\dot{Q}^1=v_{x1}=\boldsymbol{v}_1\cdot\boldsymbol{e}_x,\ldots,\dot{Q}^{3N}=v_{zN}$, we can write the Darwin Lagrangian, given by Eqs. (2)–(4), as follows:

$$L(Q,\dot{Q})=\frac{1}{2}\sum_{a,b=1}^{3N}G_{ab}(Q)\dot{Q}^a\dot{Q}^b-V(Q). \quad (8)$$

Here the diagonal elements of the metric $G_{ab}(Q)$ are given by

$$G_{11}=G_{22}=G_{33}=m_1, \quad G_{44}=m_2,\ldots,$$
$$G_{3N-1,3N-1}=G_{3N,3N}=m_N, \quad (9)$$

and the off diagonal by

$$G_{12}=G_{21}=0, \quad G_{13}=G_{31}=0, \tag{10}$$

$$G_{14}(Q)=G_{41}(Q)=\frac{q_1q_2}{2c^2}\left(\frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}+\frac{(x_1-x_2)^2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|^3}\right), \tag{11}$$

$$G_{15}(Q)=G_{51}(Q)=\frac{q_1q_2}{2c^2}\left(\frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}+\frac{(x_1-x_2)(y_1-y_2)}{|\boldsymbol{r}_1-\boldsymbol{r}_2|^3}\right), \tag{12}$$

$$G_{16}(Q)=G_{61}(Q)=\frac{q_1q_2}{2c^2}\left(\frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}+\frac{(x_1-x_2)(z_1-z_2)}{|\boldsymbol{r}_1-\boldsymbol{r}_2|^3}\right), \dots \tag{13}$$

where $|\boldsymbol{r}_1-\boldsymbol{r}_2|=\sqrt{(Q^1-Q^4)^2+(Q^2-Q^5)^2+(Q^3-Q^6)^2}$, etc., and where $V(Q)$ is simply the Coulomb potential energy. The corresponding Hamiltonian is by definition given by

$$H(Q,P)=\sum_{a=1}^{3N}\dot{Q}^aP_a-L, \tag{14}$$

the Legendre transform of L. Here the generalized momenta $P_a=\partial L/\partial\dot{Q}^a$. According to well known general results one then finds that the Hamiltonian is

$$H(Q,P)=\frac{1}{2}\sum_{a,b=1}^{3N}G^{ab}(Q)P_aP_b+V(Q), \tag{15}$$

where $G^{ab}(Q)$ is the matrix inverse of $G_{ab}(Q)$. The problem of finding the Darwin Hamiltonian is thus the problem of inverting the matrix given by Eqs. (9)–(13). A formula for the exact general inverse is easily seen to be very complicated and is unlikely to yield physically useful insight. Essén [38] approached the inversion problem through series expansion. The first term is the inverse of the diagonal mass matrix. Keeping also the next two terms gives

$$\mathcal{H}=\sum_{i=1}^{N}\left(\left[\frac{\boldsymbol{p}_i^2}{2m_i}+\frac{q_i}{2}\phi(\boldsymbol{r}_i)\right]-\frac{q_i}{2m_ic}\boldsymbol{p}_i\cdot\boldsymbol{A}_1(\boldsymbol{r}_i)+\frac{q_i^2}{2m_ic^2}\boldsymbol{A}_1^2(\boldsymbol{r}_i)\right), \tag{16}$$

where

$$\boldsymbol{A}_1(\boldsymbol{r}_i)=\sum_{j(\neq i)}^{N}\frac{q_j[\boldsymbol{p}_j+(\boldsymbol{p}_j\cdot\boldsymbol{e}_{ij})\boldsymbol{e}_{ij}]}{2m_jc|\boldsymbol{r}_i-\boldsymbol{r}_j|}. \tag{17}$$

Keeping only the first of these two terms gives the traditional Darwin Hamiltonian as found in many textbooks [22,24–26] and applications [39,41].

Unfortunately one can easily see that in macroscopic systems such an expansion will not necessarily converge (Trubnikov and Kosachev [8]). Below we will find a way of getting around this by assuming that for a homogeneous plasma a continuum approach is valid. The algebraic problem is thereby replaced by an analytic one. This was first done by Jones and Pytte [17] and their result has been confirmed by Essén [18,19].

These attempts to find more exact Darwin Hamiltonians for macroscopic systems have met with criticism (see, for example, Krizan [42], Alastuey and Appel [43]). One of the arguments used is that a more exact Hamiltonian contain terms of higher order in v/c than those neglected in the Lagrangian. This is not correct; all higher terms are of the order of $p^2/(mc)^2$, where p is a typical momentum. The speed of light, however, appears to be of higher order in the dimensionless combination $Ne^2/(mc^2R)=Nr_{\rm e}/R$, where N is the number of particles and R is a typical size of the system. Clearly the approximation of the Hamiltonian is of a completely different nature from that used to get the Lagrangian. The Darwin Lagrangian defines a dynamical system and the exact Legendre transform of this gives the corresponding system is phase space. Approximating the Legendre transform of the Lagrangian to the Hamiltonian has nothing to do with radiation.

Alastuey and Appel [43] have claimed that there are no long range effects of magnetism and that all such conclusions drawn from the Darwin Hamiltonian are wrong. Essentially they start from Coulomb matter and radiation and then claim that this approximation must be exact for long range purposes. They then seem to forget that relativistic effects, no matter how small, grow, and even diverge, with the number of particles and become important precisely when long ranges are considered. There is nothing fundamental about the split into Coulomb field plus radiation. As discussed above the fundamental split is between velocity fields and radiation (acceleration) fields. Their claims are also in contrast to, for example, the direct numerical simulations by Mehra and De Luca [7] who find long range magnetic effects from the Darwin Lagrangian. Consequently, if they vanish in the corresponding Hamiltonian formalism, this must be due to some error. Careful mathematical investigations by Kunze and Spohn [44] have also proven the correctness of the Darwin approximation for the N-body problem.

A relativistic derivation of the Darwin Hamiltonian has recently been published by Crater and Lusanna [45]. They get around some of the algebraic complexity of the matrix inversion by assuming that electric charge is an anticommuting Grassmann variable. The empirical basis for this in essentially classical plasma problems is not clear, however.

V. FINDING THE DIVERGENCE FREE VECTOR POTENTIAL

The vector potential (4) can be obtained as the solution of

$$\nabla^2\boldsymbol{A}^c=-\frac{4\pi}{c}\boldsymbol{j}_\perp, \tag{18}$$

where

$$\boldsymbol{j}_{\perp}=\boldsymbol{j}+\frac{1}{4\pi}\nabla\frac{\partial\phi}{\partial t} \tag{19}$$

is the transverse (divergence free) current density. Here the charge and current densities are assumed to be

$$\varrho(\boldsymbol{r})=\sum_{j=1}^{N} q_j\delta(\boldsymbol{r}-\boldsymbol{r}_j), \tag{20}$$

$$\boldsymbol{j}(\boldsymbol{r})=\sum_{j=1}^{N} q_j\boldsymbol{v}_j\delta(\boldsymbol{r}-\boldsymbol{r}_j). \tag{21}$$

The reason for using the transverse current as source term on the right-hand side is the use of the Coulomb gauge which requires the vector potential to be divergence free ($\boldsymbol{\nabla}\cdot\boldsymbol{A}^c=0$). If we take the divergence on both sides of Eq. (18) we see that if the source is not divergence free then neither is the solution. Use of the transverse current as source give the result (4). There is, however, a simpler method for finding the divergence free vector potential, which we present below since it will be needed later.

Start with the solution to the equation

$$\nabla^2\boldsymbol{A}=-\frac{4\pi}{c}\boldsymbol{j} \tag{22}$$

with the ordinary current Eq. (21) as source. This solution is

$$\boldsymbol{A}(\boldsymbol{r})=\sum_{j}^{N}\frac{q_j\boldsymbol{v}_j}{c|\boldsymbol{r}-\boldsymbol{r}_j|}. \tag{23}$$

We now wish to make this divergence free by adding a suitable gradient

$$\boldsymbol{A}^c=\boldsymbol{A}-\nabla\Phi \tag{24}$$

such that

$$\boldsymbol{\nabla}\cdot\boldsymbol{A}^c=\boldsymbol{\nabla}\cdot(\boldsymbol{A}-\nabla\Phi)=\boldsymbol{0}, \tag{25}$$

while the curl of $\boldsymbol{A}^c$ remains the same as the curl of $\boldsymbol{A}$.

Consider one of the terms in the sum of Eq. (23). Choose the origin at $\boldsymbol{r}_i$ and the z axis along $\boldsymbol{v}_i$. The term is then

$$\boldsymbol{A}_i=\frac{q_i v_i}{c}\frac{\boldsymbol{e}_z}{r}. \tag{26}$$

We thus need to make vector fields of type

$$\boldsymbol{A}_n=r^n\boldsymbol{e}_z \tag{27}$$

divergence free (transverse). Here $n=-1$ but the general result will be useful below. Introduce spherical coordinates, r,θ,φ, so that $\boldsymbol{e}_z=\cos\theta\,\boldsymbol{e}_r-\sin\theta\,\boldsymbol{e}_\theta$, and $\boldsymbol{e}_z\cdot\boldsymbol{e}_r=\cos\theta$. Using these one finds that $\boldsymbol{\nabla}\cdot\boldsymbol{A}_n=\boldsymbol{\nabla}\cdot(r^n\boldsymbol{e}_z)=nr^{n-1}\cos\theta$. If we now choose

$$\Phi_n=\frac{r^{n+1}\cos\theta}{n+3} \tag{28}$$

we find that the Laplacian is $\nabla^2\Phi_n=nr^{n-1}\cos\theta$. Therefore $\boldsymbol{A}_n^c=\boldsymbol{A}_n-\nabla\Phi_n$ has zero divergence. Explicitly we find

$$\boldsymbol{A}_n^c=\frac{r^n}{n+3}[(n+2)\boldsymbol{e}_z-n(\boldsymbol{e}_z\cdot\boldsymbol{e}_r)\boldsymbol{e}_r]. \tag{29}$$

For our case, $n=-1$, the relevant divergence free vector field, corresponding to $\boldsymbol{A}_i=(q_i v_i/c)r^{-1}\boldsymbol{e}_z$, is

$$\boldsymbol{A}_i^c=\frac{q_i v_i}{c}\frac{1}{2r}[\boldsymbol{e}_z+(\boldsymbol{e}_z\cdot\boldsymbol{e}_r)\boldsymbol{e}_r]. \tag{30}$$

If one now moves the origin back to $\boldsymbol{r}_i$, so that $r=|\boldsymbol{r}-\boldsymbol{r}_i|$, and use $\boldsymbol{v}_i=v_i\boldsymbol{e}_z$ we see that we end up with one of the terms in Eq. (4). The choice of Φ_{-1} is not the only possible if we only demand zero divergence, but if we also demand reasonable behavior at the origin and at infinity it becomes unique.

In summary, instead of solving Eq. (18) with divergence free source (19) we can simply solve Eq. (23) and impose zero divergence afterwards by adding a suitable gradient. A generalization of this will be useful below.

VI. VECTOR POTENTIAL IN TERMS OF CANONICAL MOMENTA

In this section we first treat the homogeneous one-component plasma. This more elementary background must be thoroughly understood before attempting the two-component plasma.

If we assume that there is only one kind of particle, with mass m and charge q, the energy of Eq. (6) becomes

$$E=\sum_{i=1}^{N}\left(\frac{\boldsymbol{p}_i^2}{2m}-\frac{q}{2mc}\boldsymbol{p}_i\cdot\boldsymbol{A}^c(\boldsymbol{r}_i)+\frac{q}{2}\phi(\boldsymbol{r}_i)\right), \tag{31}$$

where, $\boldsymbol{A}^c$ from Eq. (4),

$$\boldsymbol{A}^c(\boldsymbol{r})=\frac{q}{2c}\sum_{j}^{N}\frac{\boldsymbol{v}_j+(\boldsymbol{v}_j\cdot\boldsymbol{e}_j)\boldsymbol{e}_j}{|\boldsymbol{r}-\boldsymbol{r}_j|}, \tag{32}$$

is divergence free according to Eq. (30). In order to find the corresponding relevant expression in terms of canonical momenta, as given by Eq. (5),

$$\boldsymbol{p}_i=m\boldsymbol{v}_i+\frac{q}{c}\boldsymbol{A}^c(\boldsymbol{r}_i), \tag{33}$$

we now first replace the transverse $\boldsymbol{A}^c$ by the nontransverse $\boldsymbol{A}$ of Eq. (23),

$$\boldsymbol{A}(\boldsymbol{r})=\frac{q}{c}\sum_{j}^{N}\frac{\boldsymbol{v}_j}{|\boldsymbol{r}-\boldsymbol{r}_j|}. \tag{34}$$

We will recover a divergence free result later by adding a suitable gradient.

Now put

$$\boldsymbol{v}_i=\frac{\boldsymbol{p}_i}{m}-\frac{q}{mc}\boldsymbol{A}(\boldsymbol{r}_i) \tag{35}$$

into Eq. (34) and get

$$\boldsymbol{A}(\boldsymbol{r})=\frac{q}{c}\sum_{j}^{N}\left(\frac{\boldsymbol{p}_j}{m|\boldsymbol{r}-\boldsymbol{r}_j|}-\frac{q\boldsymbol{A}(\boldsymbol{r}_j)}{mc|\boldsymbol{r}-\boldsymbol{r}_j|}\right). \tag{36}$$

Since $\nabla^2(1/r)=-4\pi\delta(\boldsymbol{r})$ the Laplacian of both sides of this equation gives

$$\nabla^2\boldsymbol{A}(\boldsymbol{r})=-4\pi\frac{q}{c}\sum_{j}^{N}\left(\frac{\boldsymbol{p}_j}{m}\delta(\boldsymbol{r}-\boldsymbol{r}_j)-\frac{q\boldsymbol{A}(\boldsymbol{r}_j)}{mc}\delta(\boldsymbol{r}-\boldsymbol{r}_j)\right). \tag{37}$$

Denote the number density of the particles

$$n(\boldsymbol{r})=\sum_{j=1}^{N}\delta(\boldsymbol{r}-\boldsymbol{r}_j). \tag{38}$$

The second sum on the right-hand side can then be written as

$$4\pi\frac{q^2}{mc^2}\sum_{j}^{N}\boldsymbol{A}(\boldsymbol{r}_j)\delta(\boldsymbol{r}-\boldsymbol{r}_j)=4\pi\frac{q^2}{mc^2}\sum_{j}^{N}\boldsymbol{A}(\boldsymbol{r})\delta(\boldsymbol{r}-\boldsymbol{r}_j) \tag{39}$$

$$=4\pi\frac{q^2}{mc^2}\boldsymbol{A}(\boldsymbol{r})\sum_{j}^{N}\delta(\boldsymbol{r}-\boldsymbol{r}_j)$$

$$=4\pi\frac{q^2}{mc^2}\boldsymbol{A}(\boldsymbol{r})n(\boldsymbol{r}). \tag{40}$$

We now introduce the electric momentum current density

$$\boldsymbol{j}_p(\boldsymbol{r})\equiv\frac{q}{m}\sum_{j=1}^{N}\boldsymbol{p}_j\delta(\boldsymbol{r}-\boldsymbol{r}_j). \tag{41}$$

Using this, Eq. (37) can be written as

$$\left(\nabla^2-4\pi\frac{q^2}{mc^2}n(\boldsymbol{r})\right)\boldsymbol{A}(\boldsymbol{r})=-\frac{4\pi}{c}\boldsymbol{j}_p(\boldsymbol{r}). \tag{42}$$

This equation determines the vector potential in terms of the canonical momenta.

In order to solve this we must now make a *crucial approximation.* We assume that we can use an averaged, smooth, density $n(\boldsymbol{r})$ instead of the δ function sum. If this is allowed we can find an explicit solution for the case of a constant density: $n(\boldsymbol{r})=n=\text{const}$. We introduce the notation

$$r_0=\frac{q^2}{mc^2},\quad \frac{1}{\lambda^2}=\frac{q^2}{mc^2}n=r_0n, \tag{43}$$

for the classical particle length r_0 and the Yukawa damping length λ, respectively. This allows us to write Eq. (42) as

$$\left(\nabla^2-\frac{4\pi}{\lambda^2}\right)\boldsymbol{A}(\boldsymbol{r})=-\frac{4\pi}{c}\boldsymbol{j}_p(\boldsymbol{r}). \tag{44}$$

The physically interesting solution, which we denote $\boldsymbol{A}_p$, is given by

$$\boldsymbol{A}_p(\boldsymbol{r})=\frac{q}{mc}\sum_{j}^{N}\frac{\boldsymbol{p}_j\exp(-|\boldsymbol{r}-\boldsymbol{r}_j|/\lambda)}{|\boldsymbol{r}-\boldsymbol{r}_j|}. \tag{45}$$

There now remains to find the divergence free version of this vector field.

VII. DIVERGENCE FREE YUKAWA VECTOR POTENTIAL

Consider one of the terms in the sum of Eq. (45). Choose the origin at $\boldsymbol{r}_i$ and let $\boldsymbol{p}_i=p_i\boldsymbol{e}_z$. We now need to find out how to remove the divergence of a vector field

$$\boldsymbol{A}(\boldsymbol{r})=\frac{\exp(-r/\lambda)}{r}\boldsymbol{e}_z. \tag{46}$$

Considerations similar to those in Eqs. (23)–(27) above show that a good scalar function is given by

$$\Phi(\boldsymbol{r})=\frac{[\exp(r/\lambda)-(1+r/\lambda)]\exp(-r/\lambda)}{(r/\lambda)^2}\cos\theta. \tag{47}$$

The resulting zero divergence vector field can be written as

$$\begin{aligned}\boldsymbol{A}^c(\boldsymbol{r})&=\boldsymbol{A}(\boldsymbol{r})-\nabla\Phi(\boldsymbol{r})\\&=\frac{\exp(-r/\lambda)}{r}\left\{\frac{\exp(r/\lambda)-(1+r/\lambda)}{(r/\lambda)^2}\right.\\&\quad\left.\times[3(\boldsymbol{e}_z\cdot\boldsymbol{e}_r)\boldsymbol{e}_r-\boldsymbol{e}_z]+[\boldsymbol{e}_z-(\boldsymbol{e}_z\cdot\boldsymbol{e}_r)\boldsymbol{e}_r]\right\}.\end{aligned} \tag{48}$$

This field has also the property that, for λ going to infinity it goes to a field like that in Eq. (27),

$$\lim_{\lambda\to\infty}\boldsymbol{A}^c(\boldsymbol{r})=\frac{1}{2r}[\boldsymbol{e}_z+(\boldsymbol{e}_z\cdot\boldsymbol{e}_r)\boldsymbol{e}_r]. \tag{49}$$

In order to write the expression (48) in a more compact way we introduce the definitions

$$g(x)\equiv1-\frac{\exp(x)-(1+x)}{x^2},$$

and,

$$h(x)\equiv3\frac{\exp(x)-(1+x)}{x^2}-1. \tag{50}$$

They have the property that

$$\lim_{x\to0}g(x)=\lim_{x\to0}h(x)=\frac{1}{2}, \tag{51}$$

and this is the relevant limit for large values of λ, i.e., small damping. Using these we find that the expression (48) can be written as

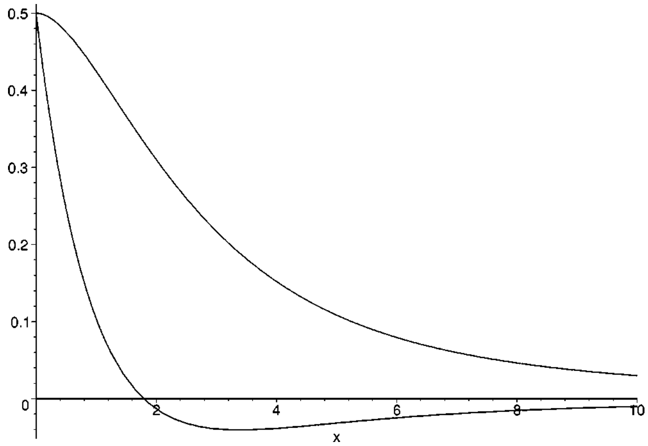

FIG. 1. Plot of the functions $\exp(-x)g(x)$ (lower curve) and $\exp(-x)h(x)$ (upper curve). These are defined in Eq. (50). The distance dependence of the magnetic interactions are $\propto\exp(-x)g(x)/x$ and $\propto\exp(-x)h(x)/x$.

$$\boldsymbol{A}^c(\boldsymbol{r})=\frac{\exp(-r/\lambda)}{r}[g(r/\lambda)\boldsymbol{e}_z+h(r/\lambda)(\boldsymbol{e}_z\cdot\boldsymbol{e}_r)\boldsymbol{e}_r]. \tag{52}$$

The qualitative distance dependencies are plotted in Fig. 1. Finally we can now write

$$\boldsymbol{A}_p^c(\boldsymbol{r})=\frac{q}{mc}\sum_j^N\frac{\exp(-|\boldsymbol{r}-\boldsymbol{r}_j|/\lambda)}{|\boldsymbol{r}-\boldsymbol{r}_j|}[g(|\boldsymbol{r}-\boldsymbol{r}_j|/\lambda)\boldsymbol{p}_j+h(|\boldsymbol{r}-\boldsymbol{r}_j|/\lambda)(\boldsymbol{p}_j\cdot\boldsymbol{e}_j)\boldsymbol{e}_j] \tag{53}$$

for the divergence free, transverse, vector potential in terms of canonical momenta.

VIII. HAMILTONIAN OF HOMOGENEOUS ONE-COMPONENT PLASMA

We have now found the Hamiltonian of a homogeneous one-component plasma. It is given by

$$H(\boldsymbol{r},\boldsymbol{p})=\sum_{i=1}^N\left(\frac{\boldsymbol{p}_i^2}{2m}-\frac{q}{2mc}\,\boldsymbol{p}_i\cdot\boldsymbol{A}_p^c(\boldsymbol{r}_i)+\frac{q}{2}\phi(\boldsymbol{r}_i)\right). \tag{54}$$

and the vector potential is given by Eq. (53) so that

$$\boldsymbol{A}_p^c(\boldsymbol{r}_i)=\frac{q}{mc}\sum_{j(\neq i)}^N\frac{\exp(-r_{ij}/\lambda)}{r_{ij}}\times[g(r_{ij}/\lambda)\boldsymbol{p}_j+h(r_{ij}/\lambda)(\boldsymbol{p}_j\cdot\boldsymbol{e}_{ij})\boldsymbol{e}_{ij}]. \tag{55}$$

The magnetic two-particle interaction energy is thus

$$E_{ij}^{\rm mag}=-\frac{q^2}{m^2c^2}\frac{\exp(-r_{ij}/\lambda)}{r_{ij}}\times[g(r_{ij}/\lambda)\boldsymbol{p}_i\cdot\boldsymbol{p}_j+h(r_{ij}/\lambda)(\boldsymbol{p}_i\cdot\boldsymbol{e}_{ij})(\boldsymbol{p}_j\cdot\boldsymbol{e}_{ij})]. \tag{56}$$

The nature of this interaction is such that the energy gets a negative contribution for parallel momentum components, $\boldsymbol{p}_i\cdot\boldsymbol{p}_j>0$, as long as the function $g(x)$ is positive, and this is the case for $0<x<1.793\,282\,133$. The energy also gets a negative contribution when the projections of the momenta along the interparticle vectors, $\boldsymbol{e}_{ij}=(\boldsymbol{r}_i-\boldsymbol{r}_j)/r_{ij}$, are parallel, $(\boldsymbol{p}_i\cdot\boldsymbol{e}_{ij})(\boldsymbol{p}_j\cdot\boldsymbol{e}_{ij})>0$, for all distances. The nature of the distance dependence of the interaction is shown in Fig. 1.

The interaction energy is clearly damped with the length scale λ. How large is λ compared to the average interparticle distance, $\overline{r_{ij}}$? From Eq. (43) we find that $\lambda=1/\sqrt{r_0 n}$. Since $n\sim 1/(\overline{r_{ij}})^3$ we find that $\lambda\sim\overline{r_{ij}}\sqrt{\overline{r_{ij}}/r_0}\gg\overline{r_{ij}}$. This is consistent with the use of the continuum approximation in the derivation of our results. A large number of particles must participate to get the screening. This screening is, however, important for the thermodynamics of this Hamiltonian to make sense. Without screening the energy diverges to minus infinity as currents correlate over larger and larger distances. This phenomenon, which Welker [46] termed "magnetische katastrophe," is thus prevented by the damping of the phase space magnetic interaction.

What is the physical significance of the results (54)–(56)? Statistical mechanics indicates that the phase space distribution of the particles in thermal equilibrium should be given by Eq. (1) and thus that the most probable configurations are those with the lowest energy. From this one concludes that the magnetic interaction should result in a correlation of particle momenta over length scales given by $\lambda=1/\sqrt{r_0 n}$, where $r_0=q^2/(mc^2)$. The question now is whether the length scales corresponding to nuclear masses m will be of importance or if the electrons with correspondingly much smaller λ determine a maximum correlation distance in real plasmas.

The idea of a one-component plasma comes mainly from solid state physics where the conduction electrons may constitute a plasma of negative particles that move in a fixed background of smeared out positive charge (a "jellium"). In a real plasma of light electrons and heavy positive ions one can expect than the length and time scales of the electron dynamics are smaller that the corresponding scales for the positive ions. The electrons also dissipate momentum much faster due to their small mass and larger Thomson cross section (the thermal, random isotropic, velocity of the electrons is of course given by the temperature, what is possibly dissipated is the net drift momentum responsible for any nonzero current density). It is therefore not unreasonable to assume that there is an approximate separation of the dynamics of the plasma so that one may consider the ions to move in background of smeared out negative charge. This positive ion one-component plasma should be characterized by λ values determined by nuclear mass scales.

IX. HAMILTONIAN OF HOMOGENEOUS TWO-COMPONENT PLASMA

Assume now that there are two kinds of particles, electrons $(-)$ and protons $(+)$, say. What happens to the screening length λ found in the one-component case? We will now investigate this question mathematically using the same formalism as for one-component plasmas.

As above we start from Eq. (22) and worry about finding the transverse vector field later. Since this equation is linear the two types of particles independently produce vector potentials,

$$\nabla^2 \boldsymbol{A}_- = -\frac{4\pi}{c}\boldsymbol{j}_-, \tag{57}$$

$$\nabla^2 \boldsymbol{A}_+ = -\frac{4\pi}{c}\boldsymbol{j}_+, \tag{58}$$

where $\boldsymbol{A}=\boldsymbol{A}_-+\boldsymbol{A}_+$ and $\boldsymbol{j}=\boldsymbol{j}_-+\boldsymbol{j}_+$. We now express the current densities Eq. (21) in terms of canonical momenta using Eq. (35). This gives

$$\boldsymbol{j}_-(\boldsymbol{r}) = \sum_{k=1}^{N_-} q_-\left(\frac{\boldsymbol{p}_k}{m_-} - \frac{q_-}{m_- c}\boldsymbol{A}(\boldsymbol{r}_k)\right)\delta(\boldsymbol{r}-\boldsymbol{r}_k)$$
$$= \boldsymbol{j}_{p-}(\boldsymbol{r}) - \frac{q_-^2}{m_- c}\boldsymbol{A}(\boldsymbol{r})n_-(\boldsymbol{r}), \tag{59}$$

$$\boldsymbol{j}_+(\boldsymbol{r}) = \sum_{l=1}^{N_+} q_+\left(\frac{\boldsymbol{p}_l}{m_+} - \frac{q_+}{m_+ c}\boldsymbol{A}(\boldsymbol{r}_l)\right)\delta(\boldsymbol{r}-\boldsymbol{r}_l)$$
$$= \boldsymbol{j}_{p+}(\boldsymbol{r}) - \frac{q_+^2}{m_+ c}\boldsymbol{A}(\boldsymbol{r})n_+(\boldsymbol{r}). \tag{60}$$

Here we have introduced the electric momentum current densities,

$$\boldsymbol{j}_p(\boldsymbol{r}) = \boldsymbol{j}_{p-}(\boldsymbol{r}) + \boldsymbol{j}_{p+}(\boldsymbol{r}) = \frac{q_-}{m_-}\sum_{k=1}^{N-}\boldsymbol{p}_k\delta(\boldsymbol{r}-\boldsymbol{r}_k)$$
$$+ \frac{q_+}{m_+}\sum_{l=1}^{N+}\boldsymbol{p}_l\delta(\boldsymbol{r}-\boldsymbol{r}_l), \tag{61}$$

of positive and negative particles, respectively, and also the number densities of negative n_- and positive n_+ particles according to Eq. (38). These obey $n=n_-+n_+$. Use of this and of $\boldsymbol{A}=\boldsymbol{A}_-+\boldsymbol{A}_+$ gives us the coupled equations

$$(\nabla^2 - 4\pi r_- n_-)\boldsymbol{A}_- = -\frac{4\pi}{c}\boldsymbol{j}_{p-} + 4\pi r_- n_- \boldsymbol{A}_+, \tag{62}$$

$$(\nabla^2 - 4\pi r_+ n_+)\boldsymbol{A}_+ = -\frac{4\pi}{c}\boldsymbol{j}_{p+} + 4\pi r_+ n_+ \boldsymbol{A}_-, \tag{63}$$

which determine the vector potentials in terms of canonical momenta. Here $r_- = q_-^2/(m_- c^2)$ is the classical particle radius for the negative particles and similarly for the positive particles. We first form the sum of the two equations to get

$$[\nabla^2 - 4\pi(r_- n_- + r_+ n_+)](\boldsymbol{A}_- + \boldsymbol{A}_+) = -\frac{4\pi}{c}(\boldsymbol{j}_{p-} + \boldsymbol{j}_{p+}). \tag{64}$$

To get $\boldsymbol{A}_-$ and $\boldsymbol{A}_+$ individually we need another equation. Divide Eq. (62) by $r_- n_-$ and Eq. (63) by $r_+ n_+$ and subtract the first from the second. This gives

$$\nabla^2\left(\frac{\boldsymbol{A}_+}{r_+ n_+} - \frac{\boldsymbol{A}_-}{r_- n_-}\right) = -\frac{4\pi}{c}\left(\frac{\boldsymbol{j}_{p+}}{r_+ n_+} - \frac{\boldsymbol{j}_{p-}}{r_- n_-}\right). \tag{65}$$

If we introduce

$$\boldsymbol{A}_p \equiv \boldsymbol{A}_+ + \boldsymbol{A}_- \quad \text{and} \quad \boldsymbol{A}_\pi \equiv \boldsymbol{A}_+ - \frac{r_+ n_+}{r_- n_-}\boldsymbol{A}_-, \tag{66}$$

we thus find that they are solutions of the equations

$$\left(\nabla^2 - \frac{4\pi}{\lambda^2}\right)\boldsymbol{A}_p = -\frac{4\pi}{c}\boldsymbol{j}_p, \tag{67}$$

$$\nabla^2 \boldsymbol{A}_\pi = -\frac{4\pi}{c}\boldsymbol{j}_\pi, \tag{68}$$

where

$$1/\lambda^2 = 1/\lambda_+^2 + 1/\lambda_-^2 = (1/c^2)(n_+ q_+^2/m_+ + n_- q_-^2/m_-) \tag{69}$$

is the screening length of $\boldsymbol{A}_p$ and where

$$\boldsymbol{j}_\pi \equiv \boldsymbol{j}_{p+} - \frac{r_+ n_+}{r_- n_-}\boldsymbol{j}_{p-} \tag{70}$$

is the source of the unscreened $\boldsymbol{A}_\pi$.

If we assume that we are dealing with, for example, a hydrogen or deuterium plasma, where the positive and negative charges are equal and opposite, $q_+=-q_-=e$, and thus $n_+=n_-=n$ (charge neutrality), we find that

$$\frac{1}{\lambda^2} = n\frac{e^2}{c^2}\left(\frac{1}{m_+} + \frac{1}{m_-}\right) \equiv n\frac{e^2}{\mu c^2} \equiv r_\mu n, \tag{71}$$

where μ is the reduced mass. This also makes it clear that

$$\boldsymbol{j}_\pi = \boldsymbol{j}_{p+} - \frac{m_-}{m_+}\boldsymbol{j}_{p-} = \frac{e}{m_+}\left(\sum_{l=1}^{N_+}\boldsymbol{p}_l\delta(\boldsymbol{r}-\boldsymbol{r}_l) + \sum_{k=1}^{N_-}\boldsymbol{p}_k\delta(\boldsymbol{r}-\boldsymbol{r}_k)\right) \tag{72}$$

simply is an electric momentum current density

$$\boldsymbol{j}_\pi = \frac{e}{m_+}\sum_{j=1}^{N}\boldsymbol{p}_j\delta(\boldsymbol{r}-\boldsymbol{r}_j) \equiv \frac{e}{m_+}\boldsymbol{\pi}(\boldsymbol{r}) \tag{73}$$

corresponding to the total momentum density $\boldsymbol{\pi}(\boldsymbol{r})$. Note that in the center of mass frame where the total momentum

$$\boldsymbol{p}_{\rm tot} = \int \boldsymbol{\pi}(\boldsymbol{r})dV \tag{74}$$

is zero ($\boldsymbol{p}_{\rm tot}=\boldsymbol{0}$), the unscreened $\boldsymbol{A}_\pi$ field will not have a monopole source.

The explicit solutions of Eqs. (67) and (68) are

$$\boldsymbol{A}_p(\boldsymbol{r}) = \frac{e}{m_+ c} \sum_l^{N_+} \frac{\boldsymbol{p}_l \exp(-|\boldsymbol{r}-\boldsymbol{r}_l|/\lambda)}{|\boldsymbol{r}-\boldsymbol{r}_l|} - \frac{e}{m_- c} \sum_k^{N_-} \frac{\boldsymbol{p}_k \exp(-|\boldsymbol{r}-\boldsymbol{r}_k|/\lambda)}{|\boldsymbol{r}-\boldsymbol{r}_k|} \tag{75}$$

and

$$\boldsymbol{A}_\pi(\boldsymbol{r}) = \frac{e}{m_+ c} \sum_j^{N} \frac{\boldsymbol{p}_j}{|\boldsymbol{r}-\boldsymbol{r}_j|}, \tag{76}$$

respectively. We also see that

$$\boldsymbol{A}_- = \left(1 + \frac{m_-}{m_+}\right)^{-1} (\boldsymbol{A}_p - \boldsymbol{A}_\pi),$$

and

$$\boldsymbol{A}_+ = \left(1 + \frac{m_-}{m_+}\right)^{-1} \left(\boldsymbol{A}_\pi + \frac{m_-}{m_+} \boldsymbol{A}_p\right), \tag{77}$$

and we have thus expressed the two vector potentials of the negative and positive particles, respectively, in terms of the momenta. As for the one-component plasma above the only approximation made is the replacement of the particle number densities with (smooth) constant densities.

From the preceding sections we also know how to make these transverse. For $\boldsymbol{A}_p$ we can use the recipe in Eq. (53) and for $\boldsymbol{A}_\pi$ Eq. (32) can be used. The full problem of the Darwin Hamiltonian of a homogeneous two-component plasma is therefore now solved.

X. RELATING THE ONE- AND TWO-COMPONENT APPROXIMATIONS

Superficially the results above are a bit puzzling from the point of view of the one-component results. If it really is correct to consider the positive ions as moving in an average negative (electronic) charge density at rest we should have recovered an effective field with long range screening, $\lambda_+ = 1/\sqrt{r_+ n_+}$, corresponding to the nuclear mass scale. Let us thus consider what happens in the two-component equations when the electronic current density is small.

If we assume that $\boldsymbol{j}_- = \boldsymbol{0}$. Then Eq. (59) gives that

$$\boldsymbol{j}_{p-}(\boldsymbol{r}) = \frac{q_-^2}{m_- c} \boldsymbol{A}(\boldsymbol{r}) n_-(\boldsymbol{r}) \tag{78}$$

and this inserted into Eq. (62) gives, using $\boldsymbol{A} = \boldsymbol{A}_- + \boldsymbol{A}_+$, that

$$\nabla^2 \boldsymbol{A}_- = \boldsymbol{0} \tag{79}$$

and therefore we must take $\boldsymbol{A}_-$ in terms of momenta, as well as in terms of velocities, as the zero vector $(\boldsymbol{A}_- = \boldsymbol{0})$. Returning to Eq. (78) we then see that

$$\boldsymbol{j}_{p-} = \frac{q_-^2}{m_- c} \boldsymbol{A}_+ n_- = c r_- n_- \boldsymbol{A}_+, \tag{80}$$

so that we can replace $\boldsymbol{j}_{p-}$ in Eqs. (64) and (65) by $c r_- n_- \boldsymbol{A}_+$. Both of these then give

$$\left(\nabla^2 - \frac{4\pi}{\lambda_+^2}\right) \boldsymbol{A}_p = -\frac{4\pi}{c} \boldsymbol{j}_{p+}, \tag{81}$$

for $\boldsymbol{A}_+ = \boldsymbol{A}_p$, and we see that the positive ion screening length λ_+ of the one-component model returns. This is consistent with Eq. (77) which implies that, if $\boldsymbol{A}_- = \boldsymbol{0}$, then $\boldsymbol{A}_p = \boldsymbol{A}_\pi$. According to Eqs. (75) and (76) this means that the assumption of $\boldsymbol{j}_- = \boldsymbol{0}$ makes the short range screened field (75) and the unscreened field (76) both equal to the long range screened field that is the solution of Eq. (81).

It is not obvious what these results mean. It would be very strange and unphysical, however, if the length scale λ_+ appeared suddenly when $\boldsymbol{j}_-$ becomes exactly zero. This indicates that it must be present somehow in the interplay between our unscreened $\boldsymbol{A}_\pi$ and our short range screened field $\boldsymbol{A}_p$. A detailed investigation of this must probably go beyond the smoothed constant density approximation used here. Nevertheless, we find in both our one- and our two-component approaches, the noncontradictory results that nuclear length scales $\lambda = 1/\sqrt{r_\mathrm{p} n}$, where r_p is the classical proton radius, $r_\mathrm{p} = e^2/(m_\mathrm{p} c^2)$, should be present in a real plasma.

XI. CONCLUSION

In this article we have reviewed the basis for the use of the Darwin approximation in plasma theory and in particular its use in plasma statistical mechanics. The main result is that a hypothetical equilibrium plasma has lower energy when currents are parallel (correlated), and the length scales over which these correlations are important have been considered from a theoretical point of view. Our results, while not entirely unambiguous, strongly indicate that length scales determined by nuclear mass can be important and present already in thermal equilibrium.

Indeed, the observed longevity and stability of currents and magnetic fields in astrophysical plasmas would be hard to understand if they were in conflict with an approach to equilibrium. The Darwin Hamiltonian does predict that the thermal equilibrium of plasmas can support currents and magnetic fields. The observed size of the relevant structures are much larger than that predicted by the electronic length scale. This empirical fact originally prompted the above investigation of the theoretical length scales of more realistic two-component plasmas.

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